Acta Mathematicae Applicatae Sinica, English Series
Vol. 24, No. 4 (2008) 649–654
DOI: 10.1007/s10255-007-7113-7
www.Applmath.com.cn

Acta Mathematica Applicatae Sinica, English Series

The Ruin Probability in the Presence of Extended Regular Variation and Optimal Investment

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Abstract Considering the classical model with risky investment, we are interested in the ruin probability that is minimized by a suitably chosen investment strategy for a capital market index. For claim sizes with common distribution of extended regular variation, starting from an integro-differential equation for the maximal survival probability, we find that the corresponding ruin probability as a function of the initial surplus is also extended regular variation.

Keywords Classical risk model, extended regular variation, optimal investment strategy, ruin probability
2000 MR Subject Classification 60J25, 91B30

1 Introduction

Calculation of ruin probabilities in the classical risk model is by now well understood. It is known that, if the claim sizes of the model have finite exponential moments, then the ruin probability decreases to zero exponentially fast as the initial surplus increases. For the case of heavy-tailed claims there also exists numerous results in the literature. When the insurance company is allowed to invest a portion of its surplus into a stock whose price is described by a geometric Brownian motion, an optimal investment strategy in the sense of minimizing the ruin probability was proposed by Hipp and Plum[3]. For the case that the distribution of claim sizes is regularly-varying tailed with index ϱ for some $\varrho < -1$, following Hipp and Plum's[3] idea of optimal investment, Gaier and Grandits[2] proved that the corresponding ruin probability considered as a function of the initial surplus is also regularly varying with the same index.

In this short note we aim to extend the result of Gaier and Grandits[2] in parallel to the case of extended regular variation.

The rest of this note consists of two sections. In Section 2 we formulate the risk model and prepare necessary preliminaries while in Section 3 we present and prove our main result.

2 The Model and Assumptions

Assume that the surplus process, $X = \{X(t), t \geq 0\}$, of the insurer without investments fulfills

$$dX(t) = cdt - dS(t), \qquad t \geq 0,$$

with initial surplus $X(0) = u$, where $c > 0$ denotes the constant premium rate, $S = \{S(t), t \geq 0\}$ denotes aggregate claims modeled by a compound Poisson process with a Poisson intensity

Manuscript received November 3, 2007. Revised March 25, 2008.
Supported by the National Natural Science Foundation of China (No. 10571167, No. 70501028) by Beijing Sustentation Fund for Elitist (Grant No. 20071D1600800421) and by National Social Science Foundation of China (Grant No. 05&ZD008).

$\lambda > 0$, and the common distribution of claim sizes is P satisfying $P(0) = 0$ with mean μ. Assume a positive safety loading, i.e. $c > \lambda\mu$. Furthermore, assume that the insurer is allowed to invest a portion of its surplus into a stock whose price process, $R = \{R(t), t \geq 0\}$, is modelled by the geometric Brownian motion

$$dR(t) = R(t)\big(adt + bdW(t)\big), \qquad t \geq 0,$$

with $R(0) = 1$, where $a > 0$ and $b > 0$ are two known parameters and $W = \{W(t), t \geq 0\}$ is a standard Brownian motion. At time t, the insurer holds $\vartheta(t)$ shares of the stock, which gives us the following evolution equation for the wealth process $V = \{V(t), t \geq 0\}$:

$$dV(t) = dX(t) + \vartheta(t)dR(t), \qquad t \geq 0,$$

with $V(0) = u$. Note that by doing so we have assumed zero interest rate. Our aim is to minimize the ruin probability

$$\Pr(V(t) < 0 \text{ for some } t \geq 0 \,|\, V(0) = u)$$

over all predictable (with respect to the filtration generated by S and W) investment strategies $\vartheta = \{\vartheta(t), t \geq 0\}$.

Denote by $\Phi(u)$ the survival probability under the optimal strategy and write $\widetilde{\Phi}(u) = \widetilde{\gamma}\Phi(u)$ with $\widetilde{\gamma}$ a suitably chosen constant such that $\widetilde{\Phi}(\infty) = 1$. It is shown in Hipp and Plum[3] that $\widetilde{\Phi}$ solves, after denoting $\lambda b^2/a^2$ and cb^2/a^2 with the symbols $\widetilde{\lambda}$ and $\widetilde{c}$, the following integro-differential equation

$$\widetilde{\Phi}''(u)\Big[-\widetilde{\lambda}\int_0^u \widetilde{\Phi}'(u-x)\overline{P}(x)dx + \widetilde{c}\big(\widetilde{\Phi}'(u) - \overline{P}(u)\big)\Big] = \frac{1}{2}\big(\widetilde{\Phi}'(u)\big)^2 \tag{1}$$

subject to the boundary conditions that $\widetilde{\Phi}'(0) = \widetilde{\Phi}(0)\lambda/c = 1$. Introducing $l(u) = \widetilde{\Phi}'(u)$, we get the following equivalent problem for l:

$$l'(u)\Big[-\widetilde{\lambda}\int_0^u l(u-x)\overline{P}(x)dx + \widetilde{c}\big(l(u) - \overline{P}(u)\big)\Big] = \frac{1}{2}\big(l(u)\big)^2 \tag{2}$$

with $l(0) = 1$. Through Equation (2), Hipp and Plum (see [3, Corollary 3.2]) showed that, if P has a locally bounded density, then Equation (1) admits a positive, strictly increasing, and strictly concave solution $\widetilde{\Phi} \in C^2(0,\infty) \cap C^1[0,\infty)$.

We shall assume that the distribution P has an extended regular variation characterized by the two-sided inequality

$$y^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{P}(xy)}{\overline{P}(x)} \leq \limsup_{x\to\infty} \frac{\overline{P}(xy)}{\overline{P}(x)} \leq y^{-\alpha}, \qquad \forall y > 1 \tag{3}$$

and for some $0 < \alpha \leq \beta < \infty$. For notational convenience we denote by $\overline{P} \in \mathrm{ERV}(-\alpha, -\beta)$ the regularity property of (3). Trivially, (3) is equivalent to

$$\upsilon^{\alpha} \leq \liminf_{x\to\infty} \frac{\overline{P}(x/\upsilon)}{\overline{P}(x)} \leq \limsup_{x\to\infty} \frac{\overline{P}(x/\upsilon)}{\overline{P}(x)} \leq \upsilon^{\beta}, \qquad \forall \upsilon > 1,$$

which implies that, for all large $x > 0$,

$$\overline{P}(x/\upsilon) \leq 2\upsilon^{\beta}\overline{P}(x), \quad \forall \upsilon > 1.$$

When $\alpha = \beta > 0$ the class $\mathrm{ERV}(-\alpha, -\beta)$ is reduced to the famous class $\mathcal{R}_{-\alpha}$ of distributions with regular variation. Therefore, the class ERV is a natural extension of the class $\mathcal{R}$. For more details of regular variation and extended regular variation, the reader is referred to the monograph Bingham et al.[1]. The following is a restatement of Lemma 3.1 of Tang et al.[5]:

Lemma 1. *Let P be a distribution belonging to the class* $\mathrm{ERV}(-\alpha, -\beta)$ *for some* $0 < \alpha \le \beta < \infty$. *Then, for any* $0 < \widetilde{\alpha}' < \alpha \le \beta < \widetilde{\beta}' < \infty$, *the distribution P has a finite moment of order $\widetilde{\alpha}'$ and the two-sided inequality*

$$c_2 x^{-\widetilde{\beta}'} \le \overline{P}(x) \le c_1 x^{-\widetilde{\alpha}'}$$

holds for any positive constants c_1 and c_2 independent of x and all large $x > 0$.

The purpose of this note is to show that, if the distribution of claim sizes belongs to the class $\mathrm{ERV}(-\alpha, -\beta)$ for some $1 < \alpha \le \beta < \infty$, then the same regularity property holds for the (minimal) ruin probability $\Psi \triangleq 1 - \Phi$ corresponding to the optimal investment strategy.

3 The Main Result and its Proof

Let us start from analyzing Equation (2). We shall testify that all the three lemmas of Gaier and Grandits[2] can be established in the current more general situation by adding certain appropriate conditions. The following first lemma shows that, if $\overline{P} \in \mathrm{ERV}(-\alpha, -\beta)$ for some $1 < \alpha \le \beta < \infty$, then the solution l of (2) decreases to zero faster than $\overline{P}$, as $u \to \infty$. This will allow us in the sequel to get rid of the convolution integral in (2).

Lemma 2. *Let* $\overline{P} \in \mathrm{ERV}(-\alpha, -\beta)$ *for some* $1 < \alpha \le \beta < \infty$ *(hence it has a finite mean). Then there is some $\varepsilon > 0$ such that*

$$\lim_{u \to \infty} \frac{u^{\varepsilon} l(u)}{\overline{P}(u)} = 0. \tag{4}$$

Proof. It follows from Lemma 1 that, for any fixed $0 < \alpha' < \widetilde{\alpha}' < \alpha \le \beta < \widetilde{\beta}' < \beta' < \infty$, as $u \to \infty$,

$$u^{\beta'} \overline{P}(u) \to \infty \quad \text{and} \quad u^{\alpha'} \overline{P}(u) \to 0. \tag{5}$$

Case 1. $\beta' < 1.5$. In this case the assertion follows directly from (5) and Conclusion (e)iii in the proof of Theorem 3.1 of Hipp and Plum[3], since our assumption implies that $\overline{P}$ has a finite integral over $[0, \infty)$. In detail, it follows from Hipp and Plum[3] that

$$\lim_{u \to \infty} u^{3/2} l(u) = 0.$$

Thus by (5), when $\beta' < 1.5$,

$$\lim_{u \to \infty} \frac{u^{3/2} l(u)}{u^{\beta'} \overline{P}(u)} = \lim_{u \to \infty} \frac{u^{3/2 - \beta'} l(u)}{\overline{P}(u)} = 0.$$

Case 2. $\beta' \ge 1.5$.
Step 1. If $\widetilde{\beta} \ge 0.98$, $\widetilde{\alpha} \ge \widetilde{\beta} + \frac{1}{2} - \frac{1}{2\widetilde{\beta}}$, and

$$\lim_{u \to \infty} u^{\widetilde{\beta}} l(u) = 0 \quad \text{and} \quad \lim_{u \to \infty} u^{\widetilde{\alpha}} \overline{P}(u) = 0$$

hold, then we have

$$\lim_{u\to\infty} u^{\widetilde{\beta}+1/2} l(u) = 0.$$

Here is the proof for this step. By defining

$$\psi[l](u) = -\widetilde{\lambda}\int_0^u l(u-x)\overline{P}(x)dx + \widetilde{c}\big(l(u) - \overline{P}(u)\big),$$

it follows from (2) that

$$\psi[l](u) = \frac{1}{2}\frac{l^2(u)}{l'(u)} = -\frac{1}{2}\frac{1}{(1/l)'(u)}.$$

We obtain, similarly as in the proof of Theorem 3.1 of Hipp and Plum[3], that

$$\begin{aligned}-u^{\widetilde{\beta}-1/2}\psi[l](u) \leq & \widetilde{\lambda}u^{\widetilde{\beta}-1/2}\int_0^{u-u^{1-1/2\widetilde{\beta}}} l(u-x)\overline{P}(x)dx + \widetilde{c}u^{\widetilde{\beta}-1/2}\overline{P}(u)\\ & + \widetilde{\lambda}u^{\widetilde{\beta}-1/2}\int_{u-u^{1-1/2\widetilde{\beta}}}^{u} l(u-x)\overline{P}(x)dx\\ \leq & \widetilde{\lambda}u^{\widetilde{\beta}-1/2} l(u^{1-1/2\widetilde{\beta}})\mu + \widetilde{\lambda}u^{\widetilde{\beta}+1/2-1/2\widetilde{\beta}}\overline{P}(u-u^{1-1/2\widetilde{\beta}}) + \widetilde{c}u^{\widetilde{\beta}-1/2}\overline{P}(u).\end{aligned}$$

According to our assumptions, all the three terms on the right-hand side of above expression tend to zero as $u\to\infty$. Hence,

$$\begin{aligned}\lim_{u\to\infty} u^{\widetilde{\beta}+1/2} l(u) =& \lim_{u\to\infty}\frac{u^{\widetilde{\beta}+1/2}}{1/l(u)}\\ =&\Big(\widetilde{\beta}+\frac{1}{2}\Big)\lim_{u\to\infty}\frac{u^{\widetilde{\beta}-1/2}}{(1/l)'(u)}\\ =&-(2\widetilde{\beta}+1)\lim_{u\to\infty} u^{\widetilde{\beta}-1/2}\psi[l](u) = 0.\end{aligned}$$

Step 2. Let

$$\alpha^* = \sup\big\{\alpha' > 0 : \lim_{u\to\infty} u^{\alpha'}\overline{P}(u) = 0\big\} \quad \text{and} \quad \beta^* = \inf\big\{\beta' > 0 : \lim_{u\to\infty} u^{\beta'}\overline{P}(u) = \infty\big\},$$

and assume that $\beta^* - \alpha^* \leq \frac{1}{2\alpha^*}$. Similarly to (5), for all $\alpha^* > 0$ we can find some $0 < \widetilde{\alpha} < \frac{1}{2\alpha^*}$ such that, as $u\to\infty$,

$$u^{\widetilde{\alpha}+\alpha^*}\overline{P}(u) \to \infty \quad \text{and} \quad u^{\widetilde{\alpha}}\overline{P}(u)\to 0. \tag{6}$$

Let the map $\mathcal{J}$ be defined by

$$\mathcal{J}(x) = \inf\{m = 0,1,\ldots \mid x - m/2 < 1.5\},$$

for all $x \geq 1.5$. We start with the assumption

$$\lim_{u\to\infty} u^{\widetilde{\alpha}+1/2\widetilde{\alpha}-\mathcal{J}(\widetilde{\alpha}+1/2\widetilde{\alpha})/2} l(u) = 0,$$

that is, we set $\widetilde{\beta} = \widetilde{\alpha}+1/2\widetilde{\alpha}-\mathcal{J}(\widetilde{\alpha}+1/2\widetilde{\alpha})/2$. This holds true by the result of Hipp and Plum[3] as mentioned in Case 1. Repeated application of Step 1 (note that we may look on $\widetilde{\beta}+1/2$ as a new '$\widetilde{\beta}$' at the second time) yields

$$\lim_{u\to\infty} u^{\widetilde{\alpha}+1/2\widetilde{\alpha}} l(u) = 0. \tag{7}$$

It follows from (6) and (7) that

$$\lim_{u\to\infty}\frac{u^{\widetilde{\alpha}+1/2\widetilde{\alpha}}l(u)}{u^{\widetilde{\alpha}+\alpha^*}\overline{P}(u)}=\lim_{u\to\infty}\frac{u^{1/2\widetilde{\alpha}-\alpha^*}l(u)}{\overline{P}(u)}=0.$$

By the choice of $\widetilde{\alpha}$ we know that $1/2\widetilde{\alpha}>\alpha^*$. The assertion of Lemma 2 is obtained.

Next we want to apply a theorem of Luxemburg[4], which deals with the asymptotic behavior of convolution integrals. As this result is formulated for the so-called admissible functions, we have to make sure that we can apply the result in our case as well. This is done in the following lemma.

Lemma 3. *Let P be as in Lemma 2. Then $\overline{P}$ is admissible in the sense of Luxemburg[4], i.e., $\overline{P}$ is continuous and strictly positive for all $u>0$ and it satisfies:*

(i) $\lim\limits_{x\to\infty}\overline{P}(x+s)/\overline{P}(x)=1$ for every $s>0$;

(ii) there exists a constant $\kappa\geq 1$ such that, for all $y>0$,

$$\max(\overline{P}(x)|y\leq x\leq 2y)\leq\kappa\overline{P}(2y).$$

Proof. As $\overline{P}$ is trivially continuous and strictly positive, we start with the proof of (i). This follows immediately from a well-known result - the uniform convergence theorem for functions of extended regular variation (see [1, Theorem 2.0.7]), which says that, if $f\in\mathrm{ERV}$, then for all $\Upsilon>1$ and for some constants $\widetilde{a}$ and $\widetilde{b}$, as $x\to\infty$, the relations

$$\{1+o(1)\}\gamma^{\widetilde{b}}\leq f(\gamma x)/f(x)\leq\{1+o(1)\}\gamma^{\widetilde{a}}$$

hold uniformly in $\gamma\in[1,\Upsilon]$.

As $\overline{P}$ is monotonically increasing, (ii) is a simple consequence of the definition of extended regular variation.

The last lemma below shows the asymptotic behavior as $u\to\infty$ of the convolution integral

$$\overline{P}*l(u)=\int_0^u\overline{P}(u-x)l(x)dx=\int_0^u l(u-x)\overline{P}(x)dx$$

in (2). In the sequel the notation $f(x)\sim g(x)$ stands for $\lim\limits_{x\to\infty}f(x)/g(x)=1$.

Lemma 4. *Let P as that defined in Lemma 2 and let l be the solution of (2), then we have*

$$\overline{P}*l(u)=\int_0^u\overline{P}(u-x)l(x)dx\sim\overline{P}(u)\int_0^\infty l(x)dx. \tag{8}$$

Proof. In virtue of Lemmas 2 and 3, the proof can be given by going along the same lines of the proof of Lemma 3.3 of Gaier and Grandits[2].

Now we are ready to formulate our main result:

Theorem 1. *Let $\overline{P}$ belong to the class $\mathrm{ERV}(-\alpha,-\beta)$ for some $1<\alpha\leq\beta<\infty$ and let Ψ be the minimal ruin probability. Then we have $\Psi\in\mathrm{ERV}(-\alpha_1,-\beta_1)$, where $\alpha_1\leq\beta_1$ are two positive constants not necessarily the same as α and β.*

Proof. Firstly note that, for functions f,g and some constant $C>0$, if $f\sim Cg$, then $f\in\mathrm{ERV}(-\alpha,-\beta)$ and $g\in\mathrm{ERV}(-\alpha,-\beta)$ are equivalent. Using this fact as well as Lemmas 2 and 4, we conclude from (2) that

$$-\frac{l^2}{l'}=\frac{1}{(1/l)'}\in\mathrm{ERV}(-\alpha,-\beta).$$

By a basic property of extended regular variation (see [1, page 67]), this gives that

$$\left(\frac{1}{l}\right)' \in \mathrm{ERV}(\beta, \alpha).$$

Applying the generalized Karamata's theorem (see [1, Section 2.6]), we have $1/l \in \mathrm{ERV}(\beta_2, \alpha_2)$, or, equivalently, $l \in \mathrm{ERV}(-\alpha_2, -\beta_2)$. Hence,

$$\frac{\widetilde{\Phi}'}{\widetilde{\gamma}} = \Phi' \in \mathrm{ERV}(-\alpha_2, -\beta_2).$$

Again applying the generalized Karamata's theorem, we finally obtain that

$$\Psi \in \mathrm{ERV}(-\alpha_1, -\beta_1).$$

This ends the proof of Theorem 1.

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